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AN ANALYTICAL INVESTIGATION OF FIN LINES WITH MAGNETIZED FERRITE SUBSTRATE

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Abstract

This paper presents an analysis of a unilateral fin line printed on a magnetized ferrite substrate. The network analysis method is applied to derive the determinantal equation. Numerical results are presented.

I. INTRODUCTION

Recently fin lines have become attractive for millimeter-wave integrated circuit application. Several papers have been published describing experimental and theoretical investigations for the various versions of fin line structures printed on dielectric substrates [1] - [6]. Realization of nonreciprocal devices in fin line techniques is also of interest in the millimeter-wave range. Beyer, et al. [7] have reported the experimental investigations of a fin line ferrite isolator.

This paper presents an analysis method of the fin line on a magnetized ferrite substrate. The method is based on the application of the network analysis techniques of electromagnetic fields [8], [9] in conjunction with the Galerkin's procedure. The determinantal equation for the propagation constant of a unilateral fin line is obtained via matrix formulation. Convergence checks are performed by increasing the number of basis functions for the representation of the aperture field. Some representative numerical results are included in the paper. The method of analysis is quite general and is applicable to other types of fin line structures containing anisotropic media.

II. DETERMINANTAL EQUATION

The unilateral fin line to be analyzed here is shown in Fig. 1, where the y-axis is chosen to be the direction of wave propagation. Since the dominant propagating mode of the fin line is similar to the TE_{10} mode of the conventional rectangular waveguide and the H-field near the slot is elliptically polarized, the ferrite slab should be magnetized in a direction parallel to the x-axis to realize efficient nonreciprocal circuits. When a ferrite sample is magnetized to saturation along the x-axis, the dyadic

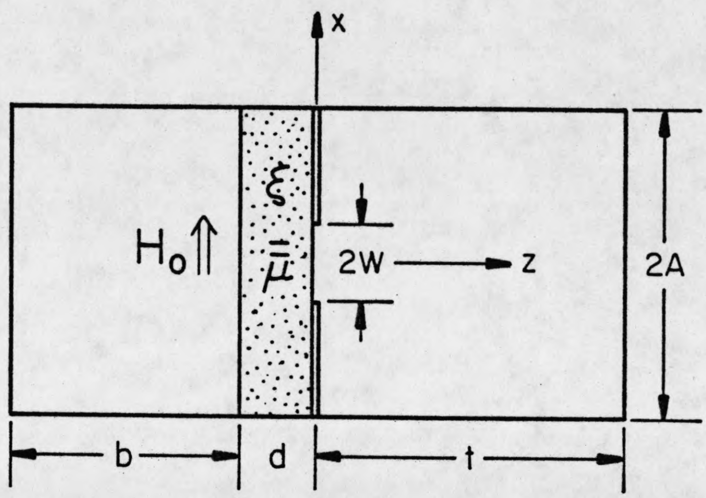

Figure 1: Cross-sectional view of unilateral fin line on ferrite substrate.

permeability of the ferrite is given by

$$\bar{\bar{\mu}} = \begin{bmatrix} \mu_0 & 0 & 0 \\ 0 & \mu & -jK \\ 0 & jK & \mu \end{bmatrix} \tag{1}$$

where μ_0 is the permeability of free space, and μ and K are dependent on the operating frequency ω, the applied dc magnetic field H_0 and the magnetization of the ferrite $4\pi M_s$.

As a first step toward deriving the determinantal equation, we express $\bar{E}_t$ and $\bar{H}_t$, the fields transverse to the z-axis, via the following Fourier integral:

$$\begin{Bmatrix} \bar{E}_t \\ \bar{H}_t \end{Bmatrix} = \frac{1}{\sqrt{2\pi}} \sum_{\ell=0}^{2} \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} \begin{Bmatrix} V_m^{(\ell)}(\beta:z)\ \bar{f}_m^{(\ell)}(\beta:x) \\ I_m^{(\ell)}(\beta:z)\ \bar{g}_m^{(\ell)}(\beta:x) \end{Bmatrix} e^{-j\beta y} d\beta \tag{2}$$

where $\ell = 1$ and $\ell = 2$ represent the E-waves ($H_z \equiv 0$) and the H-waves ($E_z \equiv 0$), respectively. The vector mode functions $\bar{f}_m^{(\ell)}$ and $\bar{g}_m^{(\ell)}$ are given in Appendix 1. They satisfy the boundary conditions at $x = \pm A$ and have the following orthonomal properties:

$$\int_{-A}^{A} \bar{f}_m^{(\ell)} \cdot \bar{f}_{m'}^{(\ell)*}\, dx = \int_{-A}^{A} \bar{g}_m^{(\ell)} \cdot \bar{g}_{m'}^{(\ell)*}\, dx = \delta_{\ell\ell'}\delta_{mm'} \tag{3}$$

where $\delta_{kk'}$ is Kronecker's delta and the asterisk denotes a complex conjugate. Also, in (2) V_m and I_m are the modal voltages and currents. The longitudinal field components are derivable from the transverse components via the relationships

$$E_z = \frac{1}{j\omega\varepsilon} \nabla \cdot (\overline{H}_t \times \hat{z}_0)$$

$$H_z = \frac{1}{j\omega\mu_{33}} [\nabla \cdot (\hat{z}_0 \times \overline{E}_t) - j\omega\, \hat{z}_0 \cdot \bar{\bar{\mu}} \cdot \overline{H}_t] \tag{4}$$

where ε is the dielectric constant of the ferrite, μ_{33} is the z-z component of $\bar{\bar{\mu}}$, which is identical to μ in the present problem, and $\hat{z}_0$ is the unit vector along the z-axis. Substituting (2) and (4) together with (1) into Maxwell's field equations, we obtain the transmission line equations for the E- and H-mode amplitudes:

$$\begin{aligned} -\frac{dV_m^{(\ell)}}{dz} &= \sum_{\ell'=1}^{x} [j\, a_m^{(\ell\ell')} I_m^{(\ell')} + b_m^{(\ell\ell')} V_m^{(\ell')}] \\ -\frac{dI_m^{(\ell)}}{dz} &= \sum_{\ell'=1}^{z} [j\, c_m^{(\ell\ell')} V_m^{(\ell')} + d_m^{(\ell\ell')} I_m^{(\ell')}] \end{aligned} \tag{5}$$

where $a_m^{(\ell\ell')}$, $b_m^{(\ell\ell')}$, $c_m^{(\ell\ell')}$ and $d_m^{(\ell\ell')}$ are given in Appendix 2. Replacement of ε by permittivity of free space ε_0, μ by μ_0, and k by 0 in (A.2) yields the transmission line equations in the air region: $0 \leq z \leq t$, $-b-d \leq z \leq -d$.

The boundary conditions to be satisfied on the z axis are expressed as

$$E_t \Big|_{z = t} = E_t \Big|_{z = -d-t} = 0 \qquad (-A \leq x \leq A) \tag{6a}$$

$$E_t \Big|_{z = -d-0} = E_t \Big|_{z = -d+0} \qquad (-A \leq x \leq A) \tag{6b}$$

$$H_t \Big|_{z = -d-0} = H_t \Big|_{z = -d+0} \qquad (-A \leq x \leq A) \tag{6c}$$

$$E_t \Big|_{z = -0} = E_t \Big|_{z = +0} \qquad (-A \leq x \leq A) \tag{6d}$$

$$H_t \Big|_{z = -0} = H_t \Big|_{z = +0} \qquad (-w \leq x \leq w) \tag{6e}$$

The above equations (6a) - (6e) can be replaced by the following continuity conditions of the modal voltages and currents:

$$V_m^{(\ell)}(t) = V_m^{(\ell)}(-b-d) = 0 \tag{7}$$

$$V_m^{(\ell)}(-d+0) = V_m^{(\ell)}(-d-0) \tag{8}$$

$$I_m^{(\ell)}(-d+0) = I_m^{(\ell)}(-d-0) \tag{9}$$

$$V_m^{(\ell)}(+0) = V_m^{(\ell)}(-0) \tag{10}$$

From (2) and (3), $V_m^{(\ell)}(0)$ is expressed in terms of the transverse electric field on the slot aperture E_0 as

$$V_m^{(\ell)}(0) = \frac{1}{\sqrt{2\pi}} \int_{-w}^{w} dx' \int_{-\infty}^{\infty} dy' \, \bar{g}_m^{(\ell)} \cdot (\hat{z}_0 \times E_0) e^{j\beta y'} \tag{11}$$

Applying the continuity conditions given by (6) to the general solutions of the transmission line equations and using the relation of (11), the modal voltages and currents are obtained in each of the subregions $0 \leq z \leq t$, $-d \leq z \leq 0$, and $-b-d \leq z \leq -d$. Substituting the results into (2) and (4), the electromagnetic fields can be expressed in terms $V_m^{(\ell)}(0)$, or implicitly, by the magnetic current distributions: $M = \hat{z}_0 \times E_0$. Application of the remaining continuity condition (6e) yields the following integral equations:

$$\sum_{\ell=1}^{2} \sum_{\ell'=1}^{2} \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} dy' \, P_m^{(\ell\ell')}(\beta) \, \bar{g}_m^{(\ell)}(\beta{:}x) \, \bar{g}_m^{(\ell')*}(\beta{:}x) \cdot \bar{M}(x',y') \, e^{-j\beta(y-y')} = 0 \tag{12}$$

where $P_m^{(\ell\ell')} = Y_m^{(\ell\ell')}(+0) - Y_m^{(\ell\ell')}(-0)$; $Y_m^{(\ell\ell')}(+0)$ and $Y_m^{(\ell\ell')}(-0)$ are the current responses of the ℓ-waves, respectively, at $z = +0$ and $z = -0$, due to the unit voltage source of the ℓ'-wave at $z = 0$ specifically,

$$Y_m^{(\ell\ell')}(\pm 0) = I_m^{(\ell)}(\pm 0)/V_m^{(\ell')}(0) . \tag{13}$$

Let β_0 be the desired propagation constant. The magnetic current may be expressed as

$$M(x',y') = M(x')\, e^{-j\beta_0 y} \tag{14}$$

Substituting (14) into (12) and integrating the result, we obtain

$$\sum_{\ell=1}^{2} \sum_{\ell'=1}^{2} \sum_{m=0}^{\infty} \int_{-w}^{w} P_m^{(\ell\ell')}(\beta_0)\, \overline{g}_m^{(\ell)}(\beta_0{:}x)\, \overline{g}_m^{(\ell')}(\beta_0{:}x') \cdot \overline{M}(x')\, dx' = 0 \tag{15}$$

The determinantal equation can be obtained by an application of the Galerkin's method to the integral equation (15). We expand the unknown magnetic current in terms of a set of known basis functions $\overline{\xi}_n$ and $\overline{\eta}_n$ as follows:

$$\overline{M}(x') = \hat{x}_0 \sum_{n'=1}^{2N_x} a_{n'} \xi_{n'}(x') + \hat{y}_0 \sum_{n'=1}^{2N_y} jb_{n'} \eta_{n'}(x') \tag{16}$$

where $\hat{x}_0$ and $\hat{y}_0$ are the unit vectors along the x and y axis, respectively. Next we substitute (16) into (15) and take the inner product of the resulting equation with $\hat{x}_0 \xi_n(x)$ and $\hat{y}_0 \eta_n(x)$. This step yields a homogeneous matrix equation for the unknown expansion coefficients a_n and b_n as

$$\sum_{n'=1}^{2N_x} F_{nn'}^{(xx)} a_{n'} + \sum_{n'=1}^{2N_y} F_{nn'}^{(xy)} b_{n'} = 0 \tag{17a}$$

$$\sum_{n'=1}^{2N_x} F_{nn'}^{(yx)} a_{n'} + \sum_{n'=1}^{2N_y} F_{nn'}^{(yy)} b_{n'} = 0 \tag{17b}$$

Equations (17a) and (17b) are valid for $n = 1, 2, \ldots, 2N_x$ and $n = 1, 2, \ldots, 2N_y$, respectively. For (17) to yield a nontrivial solution, the determinant of the coefficient matrix associated with (17) must be zero. This condition results in the determinantal equation for the propagation constant:

$$\text{DET.} \left\{ \begin{bmatrix} F^{(xx)} & F^{(xy)} \\ F^{(yx)} & F^{(yy)} \end{bmatrix} \right\} \tag{18}$$

where $F^{(xx)}$, $F^{(xy)}$, $F^{(yx)}$ and $F^{(yy)}$ are $(2N_x \times 2N_x)$, $(2N_x \times 2N_y)$, $(2N_y \times 2N_x)$ and $(2N_y \times 2N_y)$ matrices, whose elements are $F^{(xx)}_{nn'}$, $F^{(xy)}_{nn'}$, and $F^{(yy)}_{nn'}$, respectively (see Appendix 3).

The final step is the choice of the basis functions. It is desirable that the edge effect of the aperture fields be accounted for, the aperture fields be systematically improved by increasing the number of basis functions, and that the integration of (A3-b) be performed analytically. These requirements prompt us to adopt the following families of functions:

$$\begin{aligned} \xi_n &= U_n\left(\frac{x}{w}\right) \\ \eta_n &= T_{n-1}\left(\frac{x}{w}\right) \Big/ \sqrt{1-\left(\frac{x}{w}\right)^2} \end{aligned} \tag{19}$$

where T_n and U_n are Chebyshev's polynomials of the first and second kind, respectively.

III. NUMERICAL COMPUTATION

Since the fin line analyzed here is loaded symmetrically with respect to the x-axis of the ferrite substrate geometry, the (xy), (yx), (xz) and (zx) components of $\bar{\bar{\mu}}$ are zero and the field can be separated into two modes, namely, the even and odd modes which correspond to even and odd values of m in (2) or (A.3). With this separation (18) becomes

$$\text{DET.}\left\{\begin{bmatrix} F_e^{(xx)} & F_e^{(xy)} \\ F_e^{(yx)} & F_e^{(yy)} \end{bmatrix}\right\} = 0, \quad \text{DET.}\left\{\begin{bmatrix} F_0^{(xx)} & F_0^{(xy)} \\ F_0^{(yx)} & F_0^{(yy)} \end{bmatrix}\right\} = 0 \qquad (20)$$

Some results of the computation of the propagation constant for even and odd modes are given below

TABLE 1 Convergence Behavior of Phase Constant for Forward Waves

$\varepsilon_r = 12.5$ $4\pi M_s = 5000$ [Ga] $H_0 = 500$ [0_e]

$2A = t = 0.094''$ $b = 0.089''$ $d = 0.005''$

$w = 0.0235''$ freq. = 90 GHz

N ($N_x = N_y$)	Normalized propagation const. β_0/k_0	
	even	odd
1	2.026	1.585
2	1.835	1.546
3	1.834	1.546

Table 1 shows the comparison of the results for the dominant even and odd modes propagating in the positive y-direction obtained by using different expansion numbers. Note that the convergence for both the even and odd modes is quite rapid.

Figure 2 shows the dispersion characteristics of the dominant even and odd modes for a fin line on magnetized ferrite with a WR-19 waveguide shield.

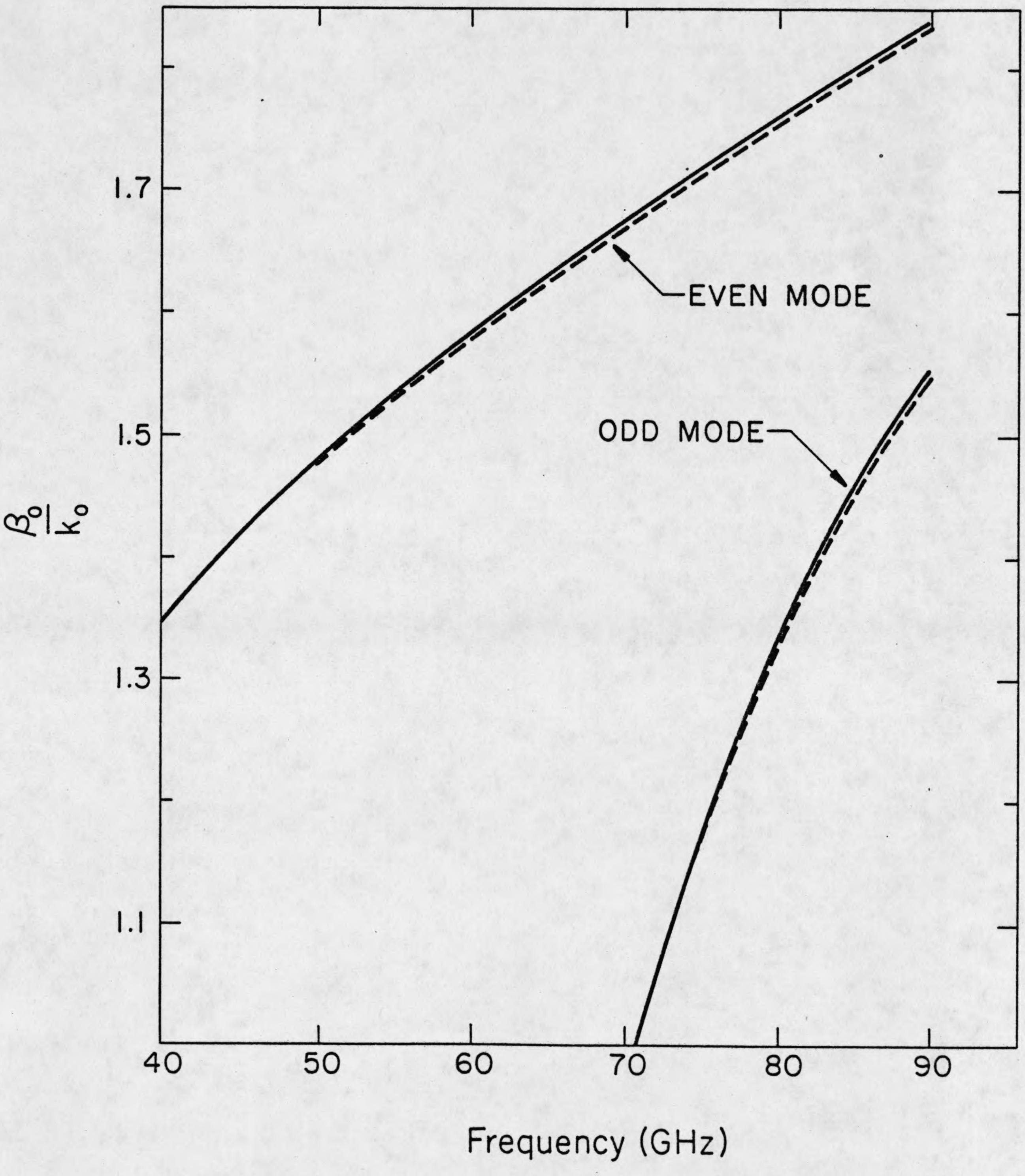

Figure 2: Dispersion characteristics of even and odd modes propagating in the positive y-direction for a fin line on ferrite and dielectric substrates ($\varepsilon_r = 12.5$, $d = 0.005''$, $2A = t - 0.094''$, $b = 0.089''$, $A/w = 2.0$, ----: $4\pi M_s = 5000$[Ga], $H_o = 500$ [O_e], ____: $K = 0$, $\mu = \mu_0$).

The characteristics of a dielectric loaded fin line in which the magnetized ferrite is replaced by an isotropic substrate ($\mu = \mu_0$, $K = 0$) are also shown in Fig. 2. In view of the convergence characteristics of the numerical results exhibited in Table 1, the number of the expansion functions (N_x, N_y) is chosen to be 2. Note that the difference between the forward and backward propagating waves for the thin ferrite substrate case is fairly small. Calculated examples for a thick ferrite substrate are shown in Fig. 3. It is observed that even for a thicker substrate the nonreciprocal effect is not substantial. In the above calculations, μ and K are assumed to be given by the following expressions

$$\begin{aligned} \frac{\mu}{\mu_0} &= \frac{\omega^2 - \gamma H_0(\gamma H_0 + \gamma 4\pi M_s)}{\omega^2 - (\gamma H_0)^2} \\ \frac{K}{\mu_0} &= \frac{\gamma 4\pi M_s \omega}{\omega^2 - (\gamma H_0)^2} \end{aligned} \tag{21}$$

where γ is the gyro-magnetic ratio.

IV. CONCLUSIONS

The unilateral fin line on a magnetized ferrite substrate has been analyzed using the network analysis technique for electromagnetic fields applied in conjunction with the Galerkin procedure. Some representative numerical solutions for the propagation constant are presented. It is found that the fin line structure containing a single ferrite substrate does not exhibit adequate nonreciprocal characteristics. Beyer, et al. [7] has reported that it is necessary to insert a spacer between the metal fin and the ferrite substrate in order to realize efficient nonreciprocal devices.

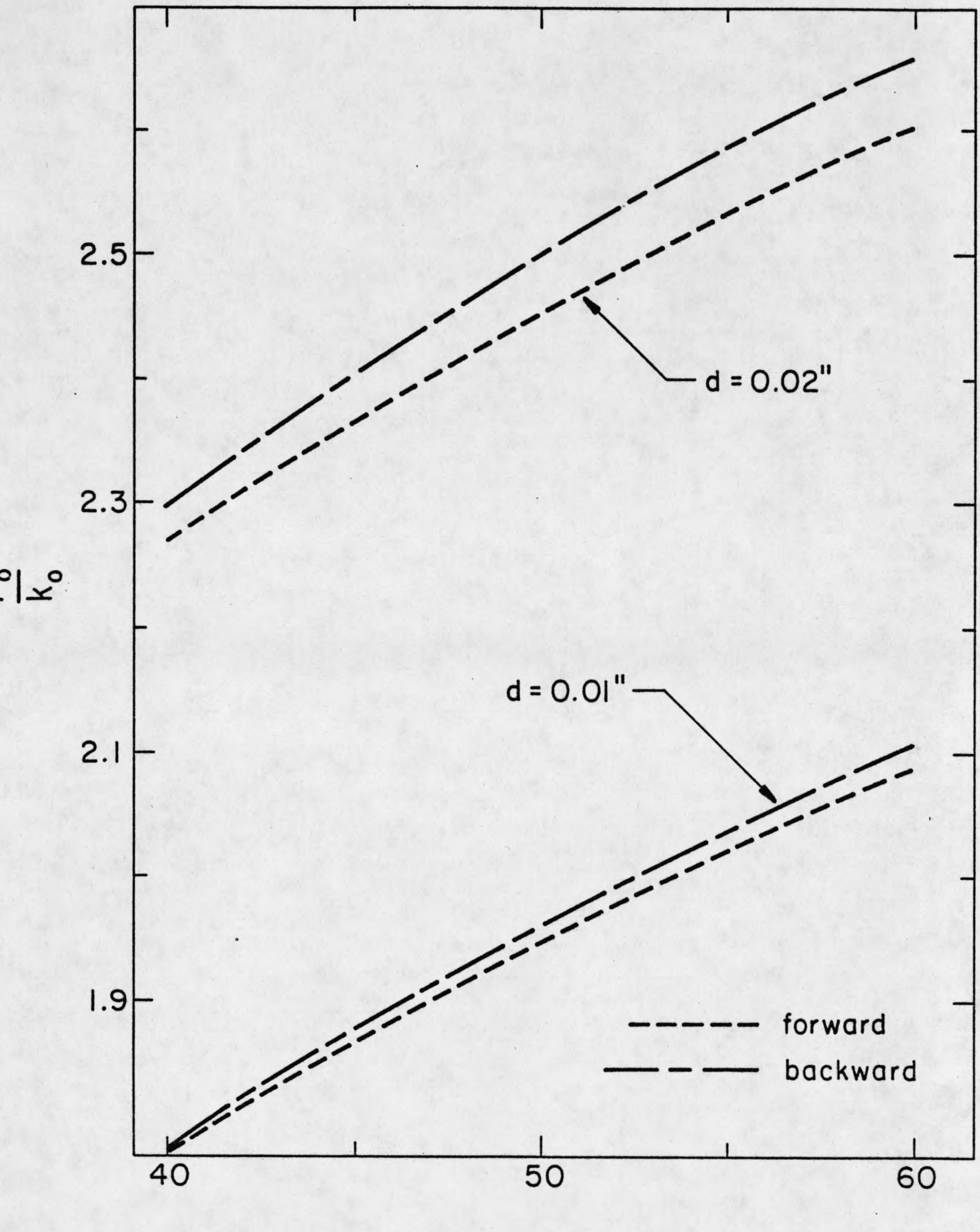

Figure 3: Dispersion characteristics of even mode propagation in the positive and negative y-directions ($\varepsilon_r = 12.5$, $4\pi M_s = 5000$[Ga], $H_0 = 500$[0_e], $2A = t = 0.094"$, $b = 0.094"$ d, $A/w = 2.0$).

The analysis of such multilayered fin lines with ferrites is beyond the scope of this paper; however, the problem is currently being investigated by the authors both theoretically and experimentally, and the results will be reported in a future publication.

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Figure 1: Cross-sectional view of unilateral fin line on ferrite substrate.

Figure 2: Dispersion characteristics of even and odd modes propagating in the positive y-direction for a fin line on ferrite and dielectric substrates (ε_r = 12.5, d = 0.005",
2A = t - 0.094", b = 0.089", A/w = 2.0,
----: $4\pi M_s$ = 5000[Ga], H_o = 500 [0_e],
____: K = 0, $\mu = \mu_0$).

Figure 3: Dispersion characteristics of even mode propagation in the positive and negative y-directions (ε_r = 12.5,
$4\pi M_s$ = 5000[Ga], H_0 = 500[0_e], 2A = t = 0.094",
b = 0.094" d, A/w = 2.0).

Appendix 1 Vector Mode Functions

$$f_m^{(1)} = \sqrt{\frac{\varepsilon_m}{2A}}\ \frac{1}{K_m}\ [\hat{x}_0\ \gamma_m \cos\ \gamma_m(y+A) - \hat{y}_0\ j\beta\ \sin\ \gamma_m(x+A)]$$

$$f_m^{(2)} = \sqrt{\frac{\varepsilon_m}{2A}}\ \frac{1}{K_m}\ [\hat{x}_0\ j\beta\ \cos\ \gamma_m(x+A) - \hat{y}_0\ \gamma_m\ \sin\ \gamma_m(x+A)]$$

$$g_m^{(\ell)} = \hat{z}_0 \times f_m^{(\ell)}$$

$$\varepsilon_m = \begin{cases} 1 & m = 0 \\ 2 & m \geq 1 \end{cases}$$

$$K_m^2 = \gamma_m^2 + \beta^2 \qquad \gamma_m = \frac{m\pi}{2A}$$

Appendix 2 Coefficients of the Transmission - Line Equation in the Ferrite Loaded Region

$$\gamma_m^{(11)} = \frac{\omega(\mu_0\beta^2 + \mu_\perp\gamma_m^2)}{K_m^2} - \frac{K_m^2}{\omega\varepsilon}$$

$$\gamma_m^{(12)} = -\gamma_m^{(21)} = -j\beta\gamma_m\,\omega(\mu_0 - \mu_\perp)/K_m^2$$

$$\gamma_m^{(22)} = \omega(\mu_0\gamma_m^2 + \mu_\perp\beta^2)/K_m^2$$

$$b_m^{(12)} = d_m^{(21)} = -j\,\frac{K}{\mu}\,\gamma_m$$

$$b_m^{(22)} = -d_m^{(22)} = -\,\frac{K}{\mu}\,\beta$$

$$c_m^{(11)} = \omega\varepsilon$$

$$c_m^{(22)} = \omega\varepsilon - \frac{K_m^2}{\omega\mu}$$

where $\mu_\perp = \mu - \dfrac{K^2}{\mu}$

Appendix 3   Elements of the Determinant

$$F^{(xx)}_{nn'} = \sum_{m=1}^{\infty} \frac{1}{AK_m^2} [\beta_0^2(Y^+_{11}-Y^-_{11})-j\beta\gamma_m(Y^-_{12}-Y^-_{21})+\gamma_m(Y^+_{22}-Y^-_{22})]\ \hat{\xi}_n\hat{\xi}_{n'}$$

$$F^{(xy)}_{nn'} = \sum_{m=1}^{\infty} \frac{1}{AK_m^2} [\beta\gamma_m(-Y^+_{11}+Y^-_{11}+Y^+_{22}-Y^-_{22})-j\beta^2Y^-_{12}-j\gamma_m^2Y^-_{21}]\ \tilde{\xi}_n\tilde{\eta}_{n'}$$

$$F^{(yx)}_{nn'} = \sum_{m=1}^{\infty} \frac{1}{AK_m^2} [\beta\gamma_m(-Y^+_{11}+Y^+_{11}+Y^+_{22}-Y^-_{22})+j\gamma_mY^-_{12}+j\beta^2Y^-_{21}]\tilde{\eta}_n\tilde{\xi}_{n'}$$

$$F^{(yy)}_{nn'} = \sum_{m=0}^{\infty} \frac{\varepsilon_m}{2AK_m^2} [\gamma_m^2\ (Y^+_{11}-Y^-_{11})+j\beta\gamma_m(Y^-_{12}-Y^-_{21})+\beta^2(Y^+_{22}-Y^-_{22})]\tilde{\eta}_n\tilde{\eta}_{n'} \qquad \text{A.3a}$$

where $Y^{\pm}_{\ell\ell'} = Y_m^{(\ell\ell')}(\pm 0)$

$$\tilde{\xi}_n = \int_{-w}^{w} \sin\gamma_m(y+a)\,\xi_n(x)\,dx$$

$$\tilde{\eta}_n = \int_{-w}^{w} \cos\gamma_m(y+a)\,\eta_n(x)\,dx \qquad \text{A.3b}$$

Note that $Y^+_{12} = Y^+_{21} = 0$